A proof of the Jacobi triple product formula

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ABSTRACT. This expository note explains a combinatorial proof of the Jacobi triple product formula which is motivated by physics.

JACOBI TRIPLE PRODUCT FORMULA (1829)

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}z)(1+q^{2n-1}z^{-1}) = \sum_{N\in\mathbb{Z}} q^{N^2}z^N$$

To prove this formula, we will follow an exposition of P. Cameron [1, §13.3] while adding some explanatory details along the way. Cameron attributes the argument to R. Borcherds.

THE DIRAC SEA

First we define the objects and notions in our toy physical model.

Define the *vacuum state* $|0\rangle$ to be the set of negative half-integers,

$$|0\rangle := \{-\tfrac{1}{2}, -\tfrac{3}{2}, -\tfrac{5}{2}, \ldots\} \subset \tfrac{1}{2}+\mathbb{Z}.$$

Define a *state* to be a subset $|\psi\rangle \subset \frac{1}{2}+\mathbb{Z}$ whose symmetric difference with $|0\rangle$ is finite. The elements of $|\psi\rangle$ which are not in $|0\rangle$ are *particles*, while the elements of $|0\rangle$ which are not in $|\psi\rangle$ are *antiparticles*.

Define the *particle number* N of a state $|\psi\rangle$ to be the number of particles minus the number of antiparticles.

Define the *energy* E of a state $|\psi\rangle$ to be

$$E = \sum\{\nu : \nu \in |\psi\rangle, \nu > 0\} - \sum\{\nu : \nu \notin |\psi\rangle, \nu < 0\} \in \tfrac{1}{2}\mathbb{Z}^{\geq 0}.$$

Note that the energy is zero if and only if $|\psi\rangle$ is the vacuum state. The number of states with energy below some bound is always finite.

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The proof

The proof will be carried out by studying the two-variable generating function which counts all possible states. We will express this generating function in two different ways. The comparison of the two resulting formulas yields the triple product formula.

Let $g(E,N)$ be the number of states with energy E and particle number N. We will index through all possible states using two formal variables q and z whose exponents record the energy and particle number. The generating function is

$$\sum_{E\in\frac{1}{2}\mathbb{Z}^{\geq 0}}\sum_{N\in\mathbb{Z}} g(E,N)q^E z^N = 1 + \left(z^{-1}+z\right)q^{\frac{1}{2}} + q + \left(z^{-1}+z\right)q^{\frac{3}{2}} + \left(z^{-2}+2+z^2\right)q^2 + \left(2z^{-1}+2z\right)q^{\frac{5}{2}} + O(q^3). \quad \text{(A)}$$

The easier way

Consider the infinite product,

$$\prod_{n=0}^{\infty}(1+q^{n+\frac{1}{2}}z).$$

When this product is expanded, the coefficient of the monomial $q^E z^N$ will equal the number of ways to obtain E as a sum of N positive half-integers. Each such combination can be associated with the unique state having no antiparticles whose particles are given by the half-integers in the combination.

We can include all states having a general combination of particles and antiparticles by using the product over z and z^{-1},

$$\prod_{n=0}^{\infty}\underbrace{(1+q^{n+\frac{1}{2}}z)}_{\text{particle with energy } n+\frac{1}{2}}\prod_{n=0}^{\infty}\overbrace{(1+q^{n+\frac{1}{2}}z^{-1})}^{\text{antiparticle with energy } n+\frac{1}{2}} = \prod_{n=0}^{\infty}(1+q^{n+\frac{1}{2}}z)(1+q^{n+\frac{1}{2}}z^{-1}).$$

We thus have the following identity of formal power series,

$$\prod_{n=0}^{\infty}(1+q^{n+\frac{1}{2}}z)(1+q^{n+\frac{1}{2}}z^{-1}) = \sum_{E\in\frac{1}{2}\mathbb{Z}^{\geq 0}}\sum_{N\in\mathbb{Z}} g(E,N)q^E z^N. \quad \text{(B)}$$

The harder way

For a given particle number $N\in\mathbb{Z}$, there is a unique state $|N\rangle$ of particle number N with the distinction of having minimal energy among all other states of particle number N. We will describe a bijection between the set of all states and the set of all pairs $(|N\rangle,\lambda)$ where $N\in\mathbb{Z}$ and λ is a partition, i.e. a monotonically decreasing sequence of nonnegative integers $\lambda_1\geq\lambda_2\geq\cdots$ which is eventually zero. This will lead to our second formula for (A).

The minimal energy state $|N\rangle$ is given by

$$|N\rangle = \{\nu \in \tfrac{1}{2} + \mathbb{Z} : \nu < N\}$$

(note that this also works for negative N). The energy of the minimal energy state $|N\rangle$ is $\frac{1}{2}N^2$.

Given a state $|\psi\rangle$, let $\overline{|\psi\rangle}$ denote the 'dual' state with the particles and antiparticles reversed (thus $\overline{|N\rangle} = |-N\rangle$). Under duals, we have $E \mapsto E$ and $N \mapsto -N$.

Consider a partition λ. We use λ to 'excite' the (anti)particles of $|N\rangle$ as follows. If N is nonnegative, we shift the largest element of $|N\rangle$ up by λ_1, shift the next largest element of $|N\rangle$ up by λ_2, and so on. If N is negative, we apply the above procedure to the dual state $|N\rangle$, and then take the dual again (this is the same as shifting 'holes' in $|N\rangle$ downward). This results in a state $|N, \lambda\rangle$ with the same particle number and energy $E = \frac{1}{2}N^2 + |\lambda|$ where $|\lambda| := \lambda_1 + \lambda_2 + \cdots$.

Conversely, suppose $|\psi\rangle$ is a state with particle number N and energy E. Suppose N is nonnegative. Let λ_1 be the distance between the largest element of $|\psi\rangle$ and the largest element of $|N\rangle$, let λ_2 be the distance between the second largest element of $|\psi\rangle$ and the second largest element of $|N\rangle$, and so on. Every λ_i is nonnegative and the sequence λ_i is monotonically decreasing, so we obtain a partition λ. Exciting the minimal energy state $|N\rangle$ according to λ recovers $|\psi\rangle$. If N is negative, then applying the above to $\overline{|\psi\rangle}$ shows that λ takes $|-N\rangle$ to $\overline{|\psi\rangle}$, hence takes $|N\rangle$ to $|\psi\rangle$.

We have thus described a bijection between the set of all states and the set of all pairs $(|N\rangle, \lambda)$ where $N \in \mathbb{Z}$ and λ is a partition. Given a pair $(|N\rangle, \lambda)$, the corresponding state has energy $E = \frac{1}{2}N^2 + |\lambda|$. Let $p(n)$ denote the number of partitions λ with $|\lambda| = n$. For any given $N \in \mathbb{Z}$, we have

$$\sum_{n=0}^{\infty} p(n) q^{\frac{1}{2}N^2+n} = \sum_{E \in \frac{1}{2}\mathbb{Z}^{\geq 0}} g(E, N) q^E.$$

By multiplying both sides by z^N and summing over $N \in \mathbb{Z}$, we obtain our second formula for our original generating function (A),

$$\left(\sum_{N\in\mathbb{Z}} q^{\frac{1}{2}N^2} z^N\right)\left(\sum_{n=0}^{\infty} p(n) q^n\right) = \sum_{E\in\frac{1}{2}\mathbb{Z}^{\geq 0}} \sum_{N\in\mathbb{Z}} g(E, N) q^E z^N. \tag{C}$$

Finishing the proof

With the help of the well-known identity for the generating function of partitions (cf. e.g. [1, (13.1.1)]),

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$

we can rewrite (C) as

$$\left(\sum_{N\in\mathbb{Z}} q^{\frac{1}{2}N^2} z^N\right)\left(\prod_{n=1}^{\infty}\frac{1}{1-q^n}\right) = \sum_{N\in\mathbb{Z}}\sum_{E\in\frac{1}{2}\mathbb{Z}^{\geq 0}} g(E,N) q^E z^N.$$

By combining this with (B), we have that

$$\prod_{n=0}^{\infty}(1+q^{n+\frac{1}{2}}z)(1+q^{n+\frac{1}{2}}z^{-1}) = \left(\sum_{N\in\mathbb{Z}} q^{\frac{1}{2}N^2} z^N\right)\left(\prod_{n=1}^{\infty}\frac{1}{1-q^n}\right).$$

Now we just reindex the left-hand side and rearrange to obtain

$$\prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}}z)(1+q^{n-\frac{1}{2}}z^{-1}) = \sum_{N\in\mathbb{Z}} q^{\frac{1}{2}N^2} z^N.$$

Replacing q with q^2 yields the triple product formula.

Macdonald Identities

The Jacobi triple product formula is one identity among many. In fact, to any "affine root system" there is an associated product identity (the "Macdonald identity" of the affine root system, cf. [3]). The Jacobi triple product formula corresponds to the "affine root system of type $\mathrm{A}_1^{(1)}$". Do Macdonald identities of other affine root systems admit similarly physical interpretations? For instance, the Macdonald identity associated to the "affine root system of type $\mathrm{A}_2^{(2)}$" is

$$\prod_{n=1}^{\infty}(1-q^n)(1-q^n z)(1-q^{n-1}z^{-1})(1-q^{2n-1}z^2)(1-q^{2n-1}z^{-2}) = \sum_{N\in\mathbb{Z}} q^{(3N^2+N)/2}(z^{3N}-z^{-3N-1}).$$

For more on this "quintuple product formula" see [2].

References

[1] P. J. Cameron. *Combinatorics: topics, techniques, algorithms.* Cambridge University Press, Cambridge, 1994.

[2] L. Carlitz and M. V. Subbarao. A simple proof of the quintuple product identity. *Proc. Amer. Math. Soc.*, 32:42–44, 1972.

[3] I. G. Macdonald. Affine root systems and Dedekind's η-function. *Invent. Math.*, 15:91–143, 1972.