An introduction to F-crystals

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0.1. **Motivation.** Let X be a smooth proper variety defined over a perfect field k of characteristic p. Let $W = W(k)$ denote the ring of Witt vectors over k and K the fraction field of W. Let σ denote the Frobenius endomorphism on W. The eigenvalues of the Frobenius operator on the cohomology groups of a smooth proper variety over a finite field are algebraic integers whose complex absolute values are determined by the Weil conjectures. The p-adic absolute values of the Frobenius eigenvalues can be studied using crystalline cohomology, and are related to the Hodge structure on the cohomology groups. For example, suppose X admits a smooth proper lift $\mathcal{X}$ to W and $k = \mathbf{F}_q$, and let $h^i = h^{i,k-i}$ denote the rank of $H^{k-i}(\mathcal{X}_K, \Omega^i_{\mathcal{X}_K/K})$. Crystalline cohomology can be used to show that if $0 = h^0 = h^1 = \cdots = h^{r-1} < h^r$ then every eigenvalue of Frobenius on $H^k(X)$ (ℓ-adic or crystalline) is divisible by q^r.

0.2. **Crystals.** Let A be a perfect k-algebra. There is an object in $(\operatorname{Spec} A/\operatorname{Spec} W)_{crys}$ which admits a map from any test object $(U \hookrightarrow T)$, namely

$$(U \hookrightarrow T, \delta) \to (\operatorname{Spec} A \to \operatorname{Spec} W(A), \gamma)$$

(this uses the universal property of Witt vectors). If F is a crystal, then $V = F(\operatorname{Spec} A \to \operatorname{Spec} W(A))$ is a p-adically complete and Hausdorff $W(A)$-module.

This extends to an equivalence of categories between F-crystals over A and pairs (V, F) consisting of a locally free $W(A)$-module V together with a σ-linear map $F\colon V \to V$ which is an isogeny, i.e. it becomes an isomorphism after inverting p. The next definition takes $A = k$.

Definition. Let V be a free and finitely generated W-module equipped with an additive endomorphism F. We say that (V, F) is a F-crystal (over k) if F is an isogeny and $F(\lambda v) = \sigma(\lambda)F(v)$ for all $v \in V$ and $\lambda \in W$. A morphism of crystals $f\colon (V, F) \to (V', F')$ is a W-linear map satisfying $fF = F'f$. Two F-crystals are isogenous if there is a morphism $f\colon (V, F) \to (V', F')$ which extends to a K-linear automorphism. An isogeny class of F-crystals is an F-isocrystal.

The absolute Frobenius endomorphism of X induces a canonical σ-linear operator F on the crystalline cohomology groups $H^i(X/W) = H^i_{crys}(X/W)$. By Poincaré duality, there is a perfect pairing (let $d = \dim X$ and $(-)' =$ quotient by torsion subgroup)

$$\langle -, - \rangle\colon H^i(X/W)' \times H^{2d-i}(X/W)' \to H^{2d}(X/W) \cong W(k)$$

which satisfies

$$\langle F(x), F(y) \rangle = p^d \sigma \langle x, y \rangle.$$

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Thus $F(x) = F(x') \implies \langle x, y\rangle = \langle x', y\rangle$ for any $y \implies x = x'$. This shows F is an isogeny of $H^i(X/W)'$, and thus $H^i(X/W)'$ is an F-crystal.

Example 1 (Exterior powers)**.** The map $(\wedge^i F)(m_1 \wedge \cdots \wedge m_i) = F(m_1) \wedge \cdots \wedge F(m_i)$ makes $(\wedge^i V, \wedge^i F)$ into an F-crystal.

Example 2 (Base change)**.** Let k' be an algebraic field extension of k and let $p^{\frac{1}{e}}$ $(e \geq 2)$ denote a formal eth root of p. Set $R = W(k')[p^{\frac{1}{e}}]$. Then $F_R = F \otimes \sigma$ is a σ-linear isogeny of $V_R = V \otimes_W R$, with σ extended to R by acting as the identity on $p^{\frac{1}{e}}$. Thus (V_R, F_R) is an F-crystal over k' of rank $e\,\mathrm{rk}_W V$.

Example 3 (Simple F-crystal)**.** For any $\lambda \in \mathbf{Q}^{\geq 0}$ let p^λ denote $(p^{\frac{1}{e}})^d$ where $\lambda = \frac{d}{e}$ with $(d, e) = 1$, $e > 0$. Set $R = W[p^\lambda]$. Extend σ to R by acting trivially on p^λ. The map

$$F\colon R \to R\colon v \mapsto p^\lambda \sigma(v)$$

is a σ-linear isogeny. This defines a simple F-crystal (over k) which is denoted E^λ.

It turns out that any simple F-crystal is, possibly after a base extension, isogenous to some E^λ. In fact, we have the following classification result for F-isocrystals.

Theorem 0.1 (Dieudonné–Manin)**.** *Suppose k is algebraically closed. Then the category of F-isocrystals is semisimple and its simple objects are the E^λ.*

Thus we may decompose any F-crystal V into simple F-isocrystals as

$$V_{W(\bar{k})} \sim \bigoplus_{i \geq 1} E^{\lambda'_i}_{W(\bar{k})}$$

for rational numbers $0 \leq \lambda'_1 \leq \lambda'_2 \leq \cdots$. Repeating each $\lambda'_i = d_i/e_i$ with multiplicity e_i, we get a sequence of rational numbers $0 \leq \lambda_1 \leq \cdots \leq \lambda_r$ called the Newton slopes of (V, F). They can be computed as follows. After a sufficiently large base change $V \mapsto V_R$ there is an R-basis for V_R with respect to which F_R is represented "in coordinates" by

$$R^r \to R^r\colon v \mapsto \begin{pmatrix} p^{\lambda_1} & * & \cdots & * \\ 0 & p^{\lambda_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p^{\lambda_r} \end{pmatrix} \sigma(v).$$

0.3. **Newton over Hodge.** Recall the theory of elementary divisors, which says there are elements $w_1, \ldots, w_r$ and $v_1, \ldots, v_r$ in V such that

$$F(v_i) = p^{a_i} w_i$$

for integers $0 \leq a_1 \leq \cdots \leq a_r$ (Hodge slopes). The Hodge numbers h^i are defined by

$$h^i = \#\text{ of times } i \text{ occurs among } \{a_1, \ldots, a_r\}.$$

We have that

$$V/F(V) \cong \bigoplus_{i \geq 0} (W/p^i W)^{h^i}.$$

Mazur [3, Theorem 2] showed these "abstract" Hodge numbers agree with classical Hodge numbers under some assumptions.

Theorem (Mazur [3]). *If X/k admits a smooth projective lift $\mathcal{X}/W$ and each Hodge cohomology group $H^j(\mathcal{X}, \Omega^i)$ is torsion-free, then $h^i(H^{i+j}(X/W)) = \mathrm{rk}_W H^j(\mathcal{X}, \Omega^i_{\mathcal{X}/W})$.*

(Note the typo on [1, p. 117].)

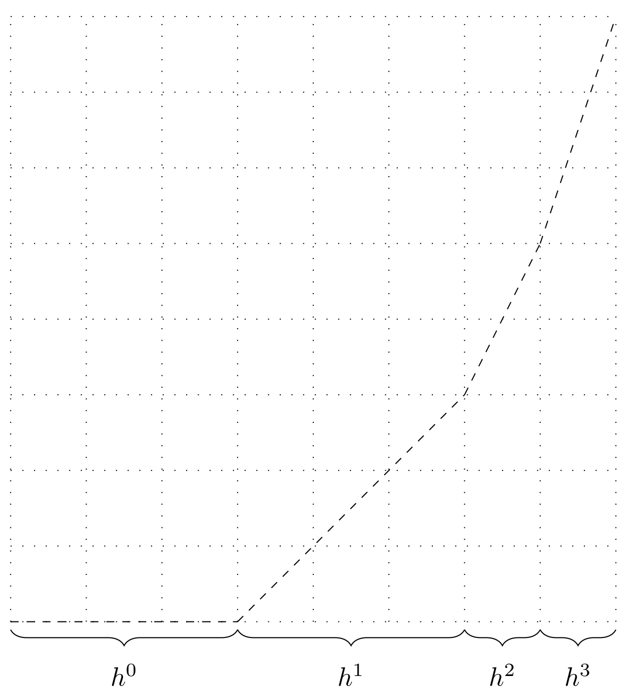
Mazur [3] also conjectured the following result which does not assume a lift exists.

Theorem 0.2 (Nygaard, Ogus). *If $H^*(X/W)$ is torsion-free and the Hodge-to-deRham spectral sequence degenerates on the first page, then $h^i(H^{i+j}(X/W)) = \mathrm{rk}_k H^j(X, \Omega^i_{X/k})$.*

The Hodge polygon of (V, F) is the graph of the Hodge function on $[0, r]$ defined on integers $0 \le i \le r$ by

$$\begin{aligned} \mathrm{Hodge}_F(i) &= \text{smallest Hodge slope of } \wedge^i V \\ &= \begin{cases} 0 & \text{if } i = 0, \\ a_1 + \cdots + a_i & \text{if } i \ge 1 \end{cases} \end{aligned}$$

and extended linearly between consecutive integers.



Remark 1 (Hodge polygon under base change). Under base change along $W \to R = W(k')[p^{\frac{1}{e}}]$, we have the $W(k')$-module isomorphism

$$V_R/F_R(V_R) = (V/F(V)) \otimes_W R \cong \bigoplus_{i \ge 0} (W(k')/p^i W(k'))^{eh^i}$$

which shows the Hodge polygon gets stretched horizontally by a factor of e.

The Hodge polygon is *not* an isogeny invariant.

We define the Newton polygon of (V, F) in the same manner using the function

$$\begin{aligned}\text{Newton}_F(i) &= \text{smallest Newton slope of } \wedge^i V \\ &= \begin{cases} 0 & \text{if } i = 0, \\ \lambda_1 + \cdots + \lambda_i & \text{if } i \geq 1. \end{cases}\end{aligned}$$

The Newton polygon is an isogeny invariant. If σ is the identity on W, then the Newton polygon of (V, F) is also the Newton polygon of the polynomial $\det(1 - Ft) \in W[t]$ but this is not generally the case if $\sigma \neq \text{id}$. The Newton polygon has the same behavior as the Hodge polygon under base change, since $E^\lambda_{W(k')[p^{\frac{1}{e}}]} \cong (E^\lambda_{W(k')})^e$ as F-crystals over k'.

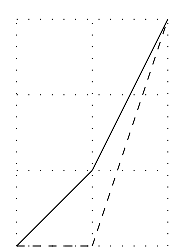
Example 4. Take $V = W \oplus W$. Let the Frobenius isogeny be

$$F = \begin{pmatrix} p & 1 \\ 0 & p^2 \end{pmatrix} \sigma.$$

The Newton slopes are $\lambda_1 = 1$ and $\lambda_2 = 2$, while

$$\begin{pmatrix} 1 & 0 \\ -p^2 & 1 \end{pmatrix} \begin{pmatrix} p & 1 \\ 0 & p^2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^3 \end{pmatrix}$$

so the Hodge slopes are $a_1 = 0$ and $a_2 = 3$.



Proposition 0.3 (Weak "Newton over Hodge")**.** *The Newton polygon of any crystal lies on or above its Hodge polygon. Both polygons have the same initial and terminal points.*

We'll show this amounts to the well-known fact that the matrix norm of an operator bounds its eigenvalues.

Proof. It suffices to prove the result after a base change, so we may assume that F is triangularizable over W. Select a norm $\|\cdot\|$ on $V[\frac{1}{p}]$ for which V is the "unit ball" and $\|pv\| = (1/p)\|v\|$ for any v. Then

$$\|F\| := \max_{v \in V} \|F(v)\| = p^{-a_1}$$

and therefore for any $i \in \{0, \ldots, r\}$,

$$\|\wedge^i F\| = p^{-\text{Hodge}_F(i)}.$$

Now suppose $F(b) = p^\lambda b$ where $\|b\| = 1$ and observe that

$$p^{-\lambda} = \|p^\lambda b\| = \|F(b)\| \leq \|F\|.$$

Applying this to $\wedge^i F$ shows that $p^{-\text{Newton}_F(i)} \leq p^{-\text{Hodge}_F(i)}$, so $\text{Newton}_F(i) \geq \text{Hodge}_F(i)$.

For the second assertion, if the crystal has rank one, then the Hodge slope and the Newton slope obviously coincide. Since the initial (resp. terminal) point of both polygons is determined by $\wedge^0 V$ (resp. $\wedge^r V$) both polygons start at $(0, 0)$ and end at $(r, \sum_i a_i)$. □

The next result is much deeper.

Theorem 0.4 ("Newton over special Hodge"; Mazur, Ogus, Nygaard)**.** *The Newton polygon of $H^k(X/W)$ lies on or above the Hodge polygon of $H^k_{dR}(X)$.*

Suppose X has a smooth proper lift $\mathcal{X}$ to W.

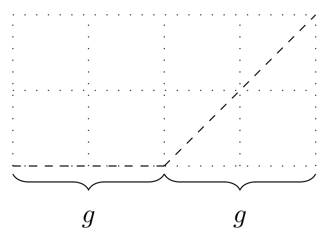
Theorem 0.5 ("Newton over generic Hodge"; Faltings, Bhatt–Morrow–Scholze)**.** *The Newton polygon of $H^k(X/W)$ lies on or above the Hodge polygon of $H^k_{dR}(\mathcal{X}_K)$.*

Also see [4] for a proof using "A_{inf}-cohomology".

0.4. **Examples.**

Definition 0.6. X is supersingular if for all n every Newton slope of $H^n(X/W)$ is $n/2$.

0.4.1. *Abelian varieties.* Let $\mathcal{A}$ be an abelian variety of relative dimension g over W and let A denote its fiber over k. Then A satisfies the hypotheses of Mazur's theorem, so $h^i(H^{i+j}(A/W))$ is the $h^{i,j}$ Hodge number of $\mathcal{A}$. This shows that $h^0(H^1(A/W)) = h^1(H^1(A/W)) = g$, so its Hodge polygon is already determined by g.



One can show (e.g. with the help of Hopf algebras) there is an isomorphism of F-crystals between the cohomology ring $H^*(A/W)$ and the exterior algebra of $H^1(A/W)$. This lets us easily compute the Newton/Hodge polygons of any $H^k(A/W)$ from $H^1(A/W)$.

Crystalline cohomology groups of abelian varieties are of particular interest because the Albanese morphism $\alpha\colon X \to \mathrm{Alb}(X)$ induces an isomorphism

$$H^1(X/W) \cong H^1(\mathrm{Alb}(X)/W).$$

Note this implies $H^1(X/W)$ is always torsion-free.

0.4.2. *An elliptic curve with CM.* The elliptic curve $E : y^2 = x^3+1$ has complex multiplication, so by a result of Serre the set of primes for which the reduction of E is supersingular has density $1/2$. One can show that E over $\mathbf{F}_p$ ($p \geq 5$) is supersingular if and only if $p \equiv 2 \pmod 3$. (Indeed the associated cusp form is

$$f(q) = \eta(6z)^4 = q\prod_{n=1}^{\infty}(1-q^{6n})^4.$$

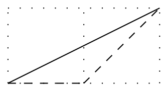
From this, if p is a prime congruent to $2, 3, 5 \pmod 6$ then the trace of F_p at p is zero.)[1]

0.4.3. *An abelian surface.* Let A denote the Jacobian of the genus two curve over $\mathbf{F}_{11}$ defined by

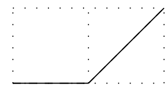
$$C : y^2 = x^6 + x^5 - x^4 - 5x^3 + x^2 + x - 1.$$

Then A is supersingular.

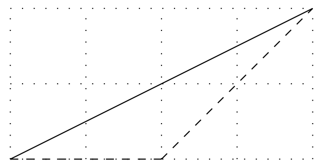
[1]In fact this is the only way that we can have supersingularity for elliptic curves on account of Hasse's bound $|a_p| \leq 2\sqrt{p}$, so if p divides a_p and $p \geq 5$ then $a_p = 0$.



(A) Supersingular $\iff$ $p \equiv 2 \pmod 3$



(B) Ordinary $\iff$ $p \equiv 1 \pmod 3$



$$H^1(C/W) = H^1(A/W)$$

0.4.4. *K3 surfaces.* Recall a smooth projective surface X/k with $H^1(X, \mathcal{O}_X) = 0$ and a trivial canonical bundle is called a K3 surface. The crystalline cohomology groups of K3 surfaces have no torsion (this can be proved using a direct calculation with Chern classes), and the Hodge-to-deRham spectral sequence degenerates at the first page.[2] By the theorem of Nygaard–Ogus, this means the "abstract" Hodge numbers of the crystalline cohomology groups can be read off from the Hodge diamond of X/k, which is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Thus only $H^2(X/W)$ is interesting.

Proposition 0.7. *If X is not supersingular then it is ordinary and there is an integer $h \in \{1, 2, \ldots, 10\}$ such that the Newton polygon of $H^2(X/W)$ has slopes*

$$(1 - (1/h), 1, 1 + (1/h))$$

with multiplicities

$$(h, 22 - 2h, h)$$

Proof. Poincaré duality says that $\langle F(x), F(y)\rangle = p^2\sigma\langle x, y\rangle$, or that

$$\langle F(x), y\rangle = p^2\sigma\langle x, F^{-1}(y)\rangle.$$

In other words, the linear isomorphism $V \to V^\vee$ determined by the cup product can be upgraded to an isomorphism of F-isocrystals

$$V \xrightarrow{\sim} V^\vee(2) := V^\vee \otimes_W E^2.$$

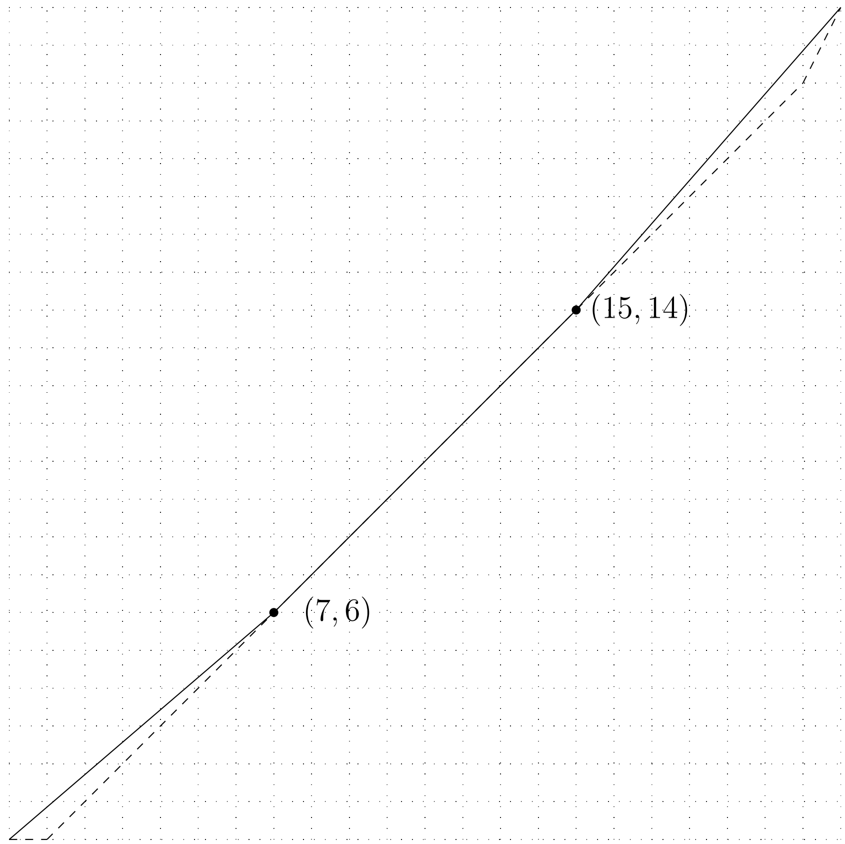
As F-isocrystals are semisimple, this shows the Newton slopes are closed under

$$\lambda \leftrightarrow 2 - \lambda.$$

Let h be the x-coordinate of the first nonzero point where the Newton polygon meets the Hodge polygon. Then the first Newton slope is $(h-1)/h = 1 - 1/h$, which implies $2 - (1 - 1/h) = 1 + 1/h$ is another; each of these occurs with multiplicity h. A priori,

[2]This can be proven using some deformation theory and a result of Deligne–Illusie, cf. [2, Prop. 2.5] for more details.

$1 \leq h \leq 11$. Since Newton lies over Hodge, the only remaining possibility is that 1 occurs as a slope $22 - 2h$ times.



Newton/Hodge polygons of a $K3$ surface, $h = 7$

Now we produce an eigenvector with slope 1 as follows. Let L be any ample line bundle on X. The pullback σ^*L of L along the absolute Frobenius $\sigma\colon X \to X$ is given by $\sigma^*L = L^p$. Thus the first Chern class $c_1(L) \in H^2(X/W)$ satisfies

$$F(c_1(L)) = c_1(\sigma^*(L)) = c_1(L^p) = pc_1(L).$$

This shows that $H^2(X/W)$ has $\lambda = 1$ as a Newton slope and so $h \neq 11$. □

In general, it is a fairly open question to determine which Newton polygons (subject to the various necessary conditions) are actually realized geometrically.

References

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[2] C. Liedtke. Lectures on supersingular K3 surfaces and the crystalline Torelli theorem. In *K3 surfaces and their moduli*, volume 315 of *Progr. Math.*, pages 171–235. Birkhäuser/Springer, [Cham], 2016.

[3] B. Mazur. Frobenius and the Hodge filtration. *Bull. Amer. Math. Soc.*, 78:653–667, 1972.

[4] Z. Yao. Frobenius and the Hodge Filtration of the Generic Fiber. *International Mathematics Research Notices*, 05 2022. rnac068.