Hodge numbers of birational Calabi–Yau varieties via p-adic Hodge theory

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1. Introduction

Today I'll explain how to use the Hodge–Tate decomposition from p-adic Hodge theory to prove the following theorem [4].[1]

Theorem A. Smooth proper varieties defined over a characteristic zero field with the same point counts over almost all finite fields have the same Hodge numbers.

The condition means X and Y can be spread out to schemes $\mathcal{X}$ and $\mathcal{Y}$ over a finitely generated subring R of the defining field such that $\#\mathcal{X}(\mathbf{F}_\ell) = \#\mathcal{Y}(\mathbf{F}_\ell)$ for all but finitely many closed points $\ell \in \operatorname{Spec} R$. Afterwards we'll prove the following theorem of Batyrev [1] using p-adic integration.

Theorem B. Smooth projective Calabi–Yau varieties over a characteristic zero field that are birational have the same point counts over almost all finite fields.

The theorems jointly prove that Hodge numbers are birational invariants for smooth projective Calabi–Yau varieties.

2. The Hodge–Tate decomposition

Let $K/\mathbf{Q}_p$ be a finite extension and let $\mathbf{C}_p$ be the completion of an algebraic closure $\overline{K}$ of K. Let $G_K = G(\overline{K}/K)$ be the absolute Galois group of K.

Theorem 2.1 (Hodge–Tate decomposition)**.** *Let X/K be a smooth proper variety. For any integer k there is an isomorphism of G_K-modules*

$$H^k_{et}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_p \cong \bigoplus_{i+j=k} H^j(X, \Omega^i_{X/K})(-i) \otimes_K \mathbf{C}_p \tag{1}$$

where $\sigma \in G_K$ acts by $\sigma \otimes \sigma$ and $H^j(X, \Omega^i_{X/K})$ has trivial G_K-action.

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[1]These notes contain nothing original. For the most part our exposition follows [4].

Computing the dimensions on both sides shows that the sum over Hodge numbers $\sum_i h^{i,k-i}$ is equal to $\dim_{\mathbf{Q}_p} H^k_{et}(X_{\overline{K}}, \mathbf{Q}_p)$ — observe that we have proven this without comparing to singular cohomology.

Note the uniformizing effect of tensoring with $\mathbf{C}_p$: the G_K-module $H^k_{et}(X_{\overline{K}}, \mathbf{Q}_p)$ can be quite complicated but becomes a direct sum of Tate twists of $\mathbf{C}_p$ after tensoring with $\mathbf{C}_p$.

An important consequence of this theorem is that individual Hodge numbers can be recovered from the G_K-module structure on p-adic etale cohomology.

Corollary 2.2 (Tate [6])**.** *Set $B_{HT} = \oplus_{i\in\mathbf{Z}} \mathbf{C}_p(i)$ (period ring). We have that*

(1) $\left(H^{i+j}_{et}(X_{\overline{K}}, \mathbf{Q}_p)(i) \otimes_{\mathbf{Q}_p} \mathbf{C}_p\right)^{G_K} \cong H^j(X, \Omega^i_{X/K})$,
(2) $\dim_K (H^k_{et}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{HT})^{G_K} = \dim_{\mathbf{Q}_p} H^k_{et}(X_{\overline{K}}, \mathbf{Q}_p)$.

Proof. By Galois theory $\mathbf{C}_p^{G_K} = K$, while for $j \neq 0$ Tate [6] showed that

$$\mathbf{C}_p(j)^{G_K} = 0. \tag{2}$$

Both claims now easily follow from the Hodge–Tate decomposition since $H^j(X, \Omega^i_{X/K})$ is seen to be the degree zero (G_K-invariant) subspace of $H^{i+j}_{et}(X_{\overline{K}}, \mathbf{Q}_p)(i) \otimes_{\mathbf{Q}_p} \mathbf{C}_p$. □

Motivated by (1) we define the ith Hodge number of a finite dimensional G_K-module V over $\mathbf{Q}_p$ to be

$$h^i V = \dim_K (V(i) \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{G_K}. \tag{3}$$

Motivated by (2) we say V is Hodge-Tate if its dimension is equal to the sum of its Hodge numbers, i.e. if

$$\dim_K (V \otimes_{\mathbf{Q}_p} B_{HT})^{G_K} = \dim_{\mathbf{Q}_p} V.$$

Tate showed that for any finite-dimensional G_K-representation V over $\mathbf{Q}_p$,

$$\dim_K (V \otimes_{\mathbf{Q}_p} B_{HT})^{G_K} \leq \dim_{\mathbf{Q}_p} V.$$

We use this to prove a few useful facts.

Lemma 2.3. *The class of Hodge–Tate representations is closed under submodules and quotients. Hodge numbers are additive on short exact sequences of Hodge–Tate representations.*

In particular, h^i factors through the Grothendieck group $K_0(HT)$ of Hodge–Tate p-adic G_K-representations. Since $[V] = [V^{ss}]$ in $K_0(HT)$ where V^{ss} is the semisimplification[2] of V, this shows that V and V^{ss} have the same Hodge numbers.

Proof. Let V be a Hodge–Tate finite dimensional G_K-representation over $\mathbf{Q}_p$. Let

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0 \tag{4}$$

[2]The direct sum of all the simple subquotients of V arising in a Jordan–Hölder series

be a short exact sequence of finite dimensional G_K-representations over $\mathbf{Q}_p$. The functor $D = D_{HT} \colon V \mapsto (V \otimes B_{HT})^{G_K}$ is left-exact which implies

$$\dim_K D(V) \le \dim_K D(V') + \dim_K D(V'') \le \dim_{\mathbf{Q}_p} V' + \dim_{\mathbf{Q}_p} V'' = \dim_{\mathbf{Q}_p} V = \dim_K D(V). \tag{5}$$

This shows that V' and V'' are Hodge–Tate. For dimension reasons D is exact:

$$0 \longrightarrow D(V') \longrightarrow D(V) \longrightarrow D(V'') \longrightarrow 0. \tag{6}$$

Each of these K-vector spaces has a grading inherited from B_{HT} and passing to the degree i components proves that $h^i V = h^i V' + h^i V''$. □

3. Proof of Theorem A

Our exposition of this proof follows Katz [3, Appendix].

Recall that we may spread out X and Y to schemes $f \colon \mathcal{X} \to \operatorname{Spec} R$ and $g \colon \mathcal{Y} \to \operatorname{Spec} R$ over a finitely generated characteristic zero ring R such that $\#\mathcal{X}(\mathbf{F}_\ell) = \#\mathcal{Y}(\mathbf{F}_\ell)$ for all but finitely many closed points $\ell \in \operatorname{Spec} R$. By inverting an element of R if needed, we may assume that each of the schemes $\mathcal{X}$ and $\mathcal{Y}$ is proper and smooth over R and that $\#\mathcal{X}(\mathbf{F}_\ell) = \#\mathcal{Y}(\mathbf{F}_\ell)$ for *all* closed points $\ell \in \operatorname{Spec} R$.

Moreover, there is a prime p such that R can be mapped injectively into the ring of integers O of a finite extension $K/\mathbf{Q}_p$.Fix such a prime p.

We will use *Frobenius automorphisms at points ℓ away from p* to show that

$$H^k_{et}(\mathcal{X}_{\overline{K}}, \mathbf{Q}_p)^{ss} \cong H^k_{et}(\mathcal{Y}_{\overline{K}}, \mathbf{Q}_p)^{ss}. \tag{7}$$

as G_K-representations. Over $\operatorname{Spec} R[1/p]$, the sheaves

$$R^k f_* \mathbf{Q}_p \quad \text{and} \quad R^k g_* \mathbf{Q}_p \tag{8}$$

are lisse and pure of weight k [2, (3.3.9)]. Pure of weight k means that for any geometric point $\overline{\ell}$ with image $\ell \in \operatorname{Spec} R[1/p]$, any eigenvalue of the geometric Frobenius automorphism $F_\ell \in G(\mathbf{F}_{\overline{\ell}}/\mathbf{F}_\ell)$ on either of the stalks $(R^k f_* \mathbf{Q}_p)_{\overline{\ell}}$ or $(R^k g_* \mathbf{Q}_p)_{\overline{\ell}}$ is an algebraic integer with complex absolute value $\sqrt{|\mathbf{F}_\ell|}^k$ for any field embedding $\overline{\mathbf{Q}_p} \to \mathbf{C}$.

By proper base change for cohomology we have isomorphisms of $G(\mathbf{F}_{\overline{\ell}}/\mathbf{F}_\ell)$-modules

$$(R^k f_* \mathbf{Q}_p)_{\overline{\ell}} \cong H^k(\mathcal{X}_{\overline{\mathbf{F}_\ell}}, \mathbf{Q}_p) \quad \text{and} \quad (R^k g_* \mathbf{Q}_p)_{\overline{\ell}} \cong H^k(\mathcal{Y}_{\overline{\mathbf{F}_\ell}}, \mathbf{Q}_p). \tag{9}$$

By the Grothendieck–Lefschetz trace formula, the automorphism F_ℓ has identical traces on the (virtual) $\pi_1(\operatorname{Spec} R[1/p])$-representations

$$\sum_{k \ge 0} (-1)^k R^k f_* \mathbf{Q}_p \quad \text{and} \quad \sum_{k \ge 0} (-1)^k R^k g_* \mathbf{Q}_p. \tag{10}$$

By purity, we can separate by weight to see that F_ℓ has identical traces on each

$$R^k f_* \mathbf{Q}_p \quad \text{and} \quad R^k g_* \mathbf{Q}_p. \tag{11}$$

By the Chebotarev density theorem, the conjugacy classes of the Frobenius elements $[F_\ell]$ equidistribute in the measure space of conjugacy classes of $\pi_1(\operatorname{Spec} R[1/p])$; in particular, they form a dense subset so these representations have the same character. Finite dimensional semisimple representations are determined by their characters so

$$(R^k f_* \mathbf{Q}_p)^{ss} \cong (R^k g_* \mathbf{Q}_p)^{ss} \quad \text{as } \pi_1(\operatorname{Spec} R[1/p])\text{-modules.} \tag{12}$$

Let $\overline{k}$ denote the geometric point corresponding to $R \to R[1/p] \to K \to \overline{K}$. Another use of proper base change shows that

$$H^k_{et}(\mathcal{X}_{\overline{K}}, \mathbf{Q}_p)^{ss} \cong (R^k f_* \mathbf{Q}_p)^{ss}_{\overline{k}} \cong (R^k g_* \mathbf{Q}_p)^{ss}_{\overline{k}} \cong H^k_{et}(\mathcal{Y}_{\overline{K}}, \mathbf{Q}_p)^{ss} \quad \text{as } G_K\text{-modules.} \tag{13}$$

Hodge numbers don't change under semisimplification so $H^k_{et}(\mathcal{X}_{\overline{K}}, \mathbf{Q}_p)$ and $H^k_{et}(\mathcal{Y}_{\overline{K}}, \mathbf{Q}_p)$ have the same Hodge numbers as G_K-modules. By the *Hodge–Tate decomposition at p* this means $\mathcal{X}_K$ and $\mathcal{Y}_K$ have the same Hodge numbers.[3]

Remark 1. The auxiliary prime p in the proof can be chosen to be arbitrarily large, in particular larger than either the dimension of X or Y or the discriminant of the "algebraic part" of R, in which case one can use an earlier theorem of Fontaine–Messing rather than the general form of the Hodge–Tate decomposition stated here.

Remark 2. There is an example[4] showing that equal point-counts for all finite fields of a single characteristic is not enough to determine Hodge numbers.

4. Example

Let $\mathbf{P}_2$ denote the projective plane and let q be a prime. Let X denote the result of blowing up $\mathbf{P}_2$ along the two points

$$(1 : \sqrt{q} : 1) \cup (1 : -\sqrt{q} : 1). \tag{14}$$

Let $\chi = \left(\frac{q}{\cdot}\right)$ denote the quadratic character associated to $\mathbf{Q}(\sqrt{q})$. Let p be another prime which is not 2 or q. If $\chi(p) = 1$, then $X(\mathbf{F}_p)$ looks like

$$\text{figure with two spirals} \tag{15}$$

and has $p^2 + 3p + 1$ $\mathbf{F}_p$-points. If $\chi(p) = -1$, then $X(\mathbf{F}_p)$ looks like

$$\text{figure with usual projective plane} \tag{16}$$

[3]Note that choosing a different embedding $R \to \mathbf{C}$ will not alter the Hodge numbers (though it will change the associated periods).

[4]`https://mathoverflow.net/questions/92958`

and has p^2+p+1 $\mathbf{F}_p$-points. The cohomology of X only differs from $\mathbf{P}_2$ in cohomological degree 2, where it becomes isomorphic to

$$H^2(\mathbf{P}_2,\mathbf{Q}_p)\oplus H^0(\operatorname{Spec}\mathbf{Q}(\sqrt{q}),\mathbf{Q}_p)(-1)=\mathbf{Q}_p(-1)\oplus(\mathbf{Q}_p\oplus\mathbf{Q}_p(\chi))(-1). \tag{17}$$

Thus the nonzero p-adic cohomology groups are given by

$$H^0=\mathbf{Q}_p,\quad H^2=\mathbf{Q}_p(-1)\oplus(\mathbf{Q}_p\oplus\mathbf{Q}_p(\chi))(-1),\quad H^4=\mathbf{Q}_p(-2) \tag{18}$$

as $G_{\mathbf{Q}_p}$-modules.

We see the second Betti number went up by two since we blew up two points, and as a Galois representation the "new" piece is $H^0(\operatorname{Spec}\mathbf{Q}(\sqrt{q}),\mathbf{Q}_p)(-1)=(\mathbf{Q}_p\oplus\mathbf{Q}_p(\chi))(-1)$.

What about the Hodge–Tate decomposition? We claim that $\mathbf{Q}_p(\chi)\otimes\mathbf{C}_p\cong\mathbf{C}_p$ as $G_{\mathbf{Q}_p}$-modules where $G_{\mathbf{Q}_p}$ acts diagonally on the left and in the usual manner on the right. Let $g_\chi=\sum_{k\in\mathbf{F}_p}\chi(k)e^{2\pi ik/p}$ be the Gauss sum of χ at p. The Gauss sum is a nonzero algebraic integer with the property that for any $\sigma\in G_{\mathbf{Q}_p}$,

$$\sigma g_\chi=\chi(\sigma)^{-1}g_\chi. \tag{19}$$

Now consider the map

$$\mathbf{Q}_p(\chi)\otimes\mathbf{C}_p\to\mathbf{C}_p \tag{20}$$

$$a\otimes x\mapsto axg_{\chi^{-1}}. \tag{21}$$

This is clearly an isomorphism of groups and for any $\sigma\in G_{\mathbf{Q}_p}$ we have

$$\sigma(a\otimes x)=\chi(\sigma)a\otimes\sigma(x)\mapsto\chi(\sigma)a\sigma(x)g_\chi=\sigma(axg_\chi), \tag{22}$$

so this is an isomorphism of G_K-modules which has "ironed out" the interesting arithmetic part of the p-adic cohomology of X.

The Hodge–Tate decomposition for the middle-degree cohomology is

$$H^2\otimes\mathbf{C}_p=\mathbf{C}_p(-2)^{\oplus 0}\oplus\mathbf{C}_p(-1)^{\oplus 3}\oplus\mathbf{C}_p(0)^{\oplus 0} \tag{23}$$

which gives the correct Hodge numbers.

Remark 3. Note the difference between χ and the p-adic cyclotomic character. By Tate's computation for $\mathbf{C}_p(1)^{G_K}$, there is no nonzero "Gauss sum" $g\in\mathbf{C}_p(1)^{G_K}$ for the p-adic cyclotomic character.

5. Proof of Theorem B

Our exposition of p-adic integration and Weil's theorem follows [5, Ch. 3].

p-adic integration. Let O be the ring of integers in K with residue field $\mathbf{F}$ of cardinality q. Let $\mathcal{X}$ be a smooth scheme over O of relative dimension d.[5] We need not assume $\mathcal{X}$ is proper. Since $\mathcal{X}$ is smooth, $\mathcal{X}(K)$ naturally has the structure of a "p-adic manifold", i.e. any point has an open neighborhood which is homeomorphic to O^d. The analytic structure on O^d carries over to $\mathcal{X}(K)$ in the usual way, giving us sheaves of analytic functions and differential forms on $\mathcal{X}(K)$.

In one respect the situation is better than for real manifolds. For one there is a canonical open compact submanifold $\mathcal{X}(O) \subset \mathcal{X}(K)$. Now consider a bianalytic map $F\colon O^d \to O^d$. Since

$$Jac(F \circ F^{-1}) = 1 = Jac(F)Jac(F^{-1}) \tag{24}$$

its Jacobian is valued in $O^\times$, i.e. $|Jac(F)| = 1$ for a change of variables F. This will imply the existence of a *canonical* measure on the canonical compact submanifold $\mathcal{X}(O)$.

The first step is to assign measures to d-forms. Let $U \subset \mathcal{X}(K)$ be an open subset equipped with a chart $\varphi\colon U \xrightarrow{\sim} O^d$. If ω is a regular d-form on U then

$$\omega \circ \varphi^{-1} = g(x)\, dx_1 \wedge \cdots \wedge dx_d \tag{25}$$

for an analytic function g on U, and we define

$$\int_U |\omega| := \int_{O^d} |\omega \circ \varphi^{-1}| = \int_{O^d} |g(x)|\, d\mu_{\text{Haar}} \tag{26}$$

where μ_{Haar} is the normalized Haar measure on O^d. This integral is independent of the choice of φ since $|Jac(\varphi' \circ \varphi^{-1})| = 1$ for another chart φ'. These local measures glue to form a measure μ_ω on $\mathcal{X}(K)$ which only depends on ω. We have proven that:

Any regular d-form ω on $\mathcal{X}(K)$ induces a Borel measure μ_ω on $\mathcal{X}(K)$ where the measure of an analytic function f on a compact open subset U is

$$\int_U |f|\, d\mu_\omega := \int_U |f\omega|. \tag{27}$$

A canonical p-adic measure. Now we use these measures to construct a canonical measure on $\mathcal{X}(O)$. The idea is to locally trivialize $\Omega^d_{\mathcal{X}/O}$ (not just $\Omega^d_{\mathcal{X}_K/K}$!) and glue the resulting measures over $\mathcal{X}(O)$.

Theorem 5.1 (Weil [8])**.**

(1) *There is a canonical Borel measure μ_{can} on $\mathcal{X}(O)$ with the property that for any Zariski open subset $\mathcal{Y} \subset \mathcal{X}$ and any regular nonvanishing d-form ω on $\mathcal{Y}/O$, μ_{can} and μ_ω agree for any analytic function f on a compact open subset $U \subset \mathcal{Y}(O)$.*

[5] We say $X \to S$ is smooth of relative dimension d if $X \to S$ is smooth and all non-empty fibers are equi-dimensional of dimension d.

(2) *The canonical measure of* $\mathcal{X}(O)$ *is*

$$\mu_{\text{can}}(\mathcal{X}(O)) = \frac{|\mathcal{X}(\mathbf{F})|}{q^d}. \tag{28}$$

Trivializing $\Omega^d_{\mathcal{X}/O}$ over a Zariski open $\mathcal{Y} \subset \mathcal{X}$ means finding a d-form ω which is nonvanishing on $\mathcal{Y}/O$. In particular, the restriction of ω to the special fiber of $\mathcal{Y}$ must be nonvanishing, so locally on $\mathcal{Y}(O)$ the form ω looks like

$$g(y)\, dy_1 \wedge \cdots \wedge dy_d \tag{29}$$

where $|g(y)| = 1$. (Although g depends on the choice of chart the condition $|g| = 1$ does not by the change of variables formula.)

Proof. Cover $\mathcal{X}$ by Zariski open subsets which are small enough to trivialize the canonical bundle $\Omega^d_{\mathcal{X}/O}$. Let $\mathcal{Y} \subset \mathcal{X}$ be an open subset in the cover and let ω be a nonvanishing d-form on $\mathcal{Y}/O$. By the construction above we have the Borel measure $\mu_{\mathcal{Y},\omega}$ on $\mathcal{Y}(K)$. If ω' is another nonvanishing d-form on $\mathcal{Y}/O$, then $\mu_{\mathcal{Y},\omega'} = \mu_{\mathcal{Y},\omega}$ on $\mathcal{Y}(O)$ since $\frac{\omega'}{\omega}$ is $O^\times$-valued on $\mathcal{Y}(O)$. Thus we get a canonical measure $\mu_{\mathcal{Y},\text{can}}$ on $\mathcal{Y}(O)$. For the same reason, these canonical measures $\mu_{\mathcal{Y},\text{can}}$ agree on overlaps of Zariski open subsets in the trivializing cover, so they glue to a canonical measure μ_{can} on $\mathcal{X}(O) = \cup_{\mathcal{Y}} \mathcal{Y}(O)$ which will satisfy (1) by construction.

To compute (2), a generalization of Hensel's lemma implies that any set-theoretic section of the reduction map $r\colon \mathcal{X}(O) \to \mathcal{X}(\mathbf{F})$ induces a homeomorphism

$$\mathcal{X}(O) \cong \mathcal{X}(\mathbf{F}) \times \mathfrak{m}^d. \tag{30}$$

This homeomorphism identifies μ_ω on the left with the product of counting measure on $\mathcal{X}(\mathbf{F})$ and Haar measure on the right. Thus for any $x_0 \in \mathcal{X}(\mathbf{F})$ we have

$$\mu_{\mathcal{X},\omega}(r^{-1}(x_0)) = \int_{\{x_0\}\times\mathfrak{m}^d} |g(x)\, dx_1 \wedge \cdots \wedge dx_d| = \mu_{\text{Haar}}(\{x_0\} \times \mathfrak{m}^d) = q^{-d}. \tag{31}$$

By (1) this is also $\mu_{\text{can}}(r^{-1}(x_0))$. Summing over $x_0 \in \mathcal{X}(\mathbf{F})$ proves the formula. □

Proof of Theorem B. Spread out to smooth proper schemes $\mathcal{X}$ and $\mathcal{Y}$ of a fixed relative dimension over a finitely generated characteristic zero ring R.For all but finitely many closed points $\ell \in \operatorname{Spec} R$ there is an embedding $R \to O = W(\mathbf{F}_\ell)$ such that $\mathcal{X}/O$ and $\mathcal{Y}/O$ have trivial canonical bundles.

Consider the diagram

$$\begin{tikzcd}
 & \mathcal{Z} \arrow[d] \arrow[ddl, bend right, "\psi"'] \arrow[ddr, bend left, "\phi"] & \\
 & \overline{\Gamma_b} \arrow[dl] \arrow[dr] & \\
\mathcal{X} \arrow[rr, dashed, "b"] & & \mathcal{Y}
\end{tikzcd} \tag{32}$$

where b is a birational equivalence, $\mathcal{Z}$ is a resolution of singularities of the Zariski closure of the graph $\Gamma_b \subset X \times Y$ of b, and ψ and ϕ are the compositions with the projection maps. Since both canonical bundles are trivial,

$$\psi^*\Omega^d_{\mathcal{X}/O} \cong \phi^*\Omega^d_{\mathcal{Y}/O} \tag{33}$$

(we say $\mathcal{X}$ and $\mathcal{Y}$ are K-equivalent). In other words, there are nonvanishing d-forms $\omega_{\mathcal{X}}$ and $\omega_{\mathcal{Y}}$ such that

$$\psi^*\omega_{\mathcal{X}} = \psi^*\omega_{\mathcal{Y}}. \tag{34}$$

Since $\mu_{\omega_{\mathcal{X}}} = \mu_{\text{can}}$ for a nonvanishing d-form by construction,

$$\mu_{\mathcal{X},\text{can}}(\mathcal{X}(O)) = \mu_{\mathcal{Z},\psi^*\omega_{\mathcal{X}}}(\mathcal{Z}(O)) = \mu_{\mathcal{Y},\text{can}}(\mathcal{Y}(O)). \tag{35}$$

This proves that $\#\mathcal{X}(\mathbf{F}) = \#\mathcal{Y}(\mathbf{F})$ by Weil's formula.

Remark 4. The proof shows that K-equivalent smooth projective varieties have the same point counts over almost all finite fields, so the restriction to birational Calabi–Yau varieties is not really the natural context for this application of p-adic Hodge theory. In fact Wang [7] has conjectured that K-equivalent smooth projective complex varieties have isomorphic Chow motives, which would give a natural explanation for several of the results we have discussed.

References

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