ON TRACE-ONE GENERATORS OF TAMELY RAMIFIED ABELIAN CUBIC FIELDS

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ABSTRACT. Let K be a tamely ramified abelian cubic number field. We prove that the number of trace-one monic integral polynomials with root field K and a given toric height is independent of K, and equal to the number of integral ideals in the quadratic cyclotomic field $\mathbb{Q}(\sqrt{-3})$ with a given norm.

1. Introduction

Our recent work [1] found a new connection between cubic abelian number fields and the arithmetic of the quadratic cyclotomic field $\mathbb{Q}(\sqrt{-3})$. This connection exists because the multiplicative group of $\mathbb{Q}(\sqrt{-3})$ is a factor of the unit group in the group algebra of the finite group $\mathbb{Z}/3\mathbb{Z}$. This note describes another facet of this connection.

Fix an abelian cubic number field K which is tamely ramified over $\mathbb{Q}$ and let F_K denote the set of polynomials of the form $t^3 - t^2 + at + b \in \mathbb{Z}[t]$ whose associated root field is isomorphic to K. For such polynomials we necessarily have $a \leq 0$ and we set $H(t^3 - t^2 + at + b) = \sqrt{1 - 3a}$ ("toric height"). The main result of this note is the following formula.

Theorem. *Let D_K denote the discriminant of K. We have that*

$$\sum_{f \in F_K} H(f)^{-2s} = D_K^{-s}(1 - 3^{-s})\zeta_{\mathbb{Q}(\sqrt{-3})}(s).$$

Remarkably, we conclude that the number of trace-one monic polynomials with a given linear coefficient that generate K is *essentially independent of K and only depends on the arithmetic of* $\mathbb{Q}(\sqrt{-3})$. This is illustrated in the table below of all trace-one monic integral cubic polynomials with $H(f) \leq 25$ which generate either $K_{49} = \mathbb{Q}(\zeta_7)^+$ or $K_{169} = \mathbb{Q}[t]/(t^3 - t^2 - 4t - 1)$.

$H(f)^2$	$f : K_f = K_{49}$
7×1	$t^3 - t^2 - 2t + 1$
7×4	$t^3 - t^2 - 9t + 1$
7×7	$t^3 - t^2 - 16t + 29,\ t^3 - t^2 - 16t - 13$
7×13	$t^3 - t^2 - 30t + 43,\ t^3 - t^2 - 30t - 41$
7×16	$t^3 - t^2 - 37t + 29$
7×19	$t^3 - t^2 - 44t + 127,\ t^3 - t^2 - 44t - 83$
7×25	$t^3 - t^2 - 58t - 13$
7×28	$t^3 - t^2 - 65t + 169,\ t^3 - t^2 - 65t - 167$
7×31	$t^3 - t^2 - 72t + 169,\ t^3 - t^2 - 72t - 41$
7×37	$t^3 - t^2 - 86t + 337,\ t^3 - t^2 - 86t - 251$
7×43	$t^3 - t^2 - 100t + 113,\ t^3 - t^2 - 100t - 181$

$H(f)^2$	$f : K_f = K_{169}$
13×1	$t^3 - t^2 - 4t - 1$
13×4	$t^3 - t^2 - 17t + 25$
13×7	$t^3 - t^2 - 30t + 25,\ t^3 - t^2 - 30t - 53$
13×13	$t^3 - t^2 - 56t + 181,\ t^3 - t^2 - 56t + 25$
13×16	$t^3 - t^2 - 69t - 131$
13×19	$t^3 - t^2 - 82t + 155,\ t^3 - t^2 - 82t - 235$
13×25	$t^3 - t^2 - 108t + 337$
13×28	$t^3 - t^2 - 121t + 545,\ t^3 - t^2 - 121t - 79$
13×31	$t^3 - t^2 - 134t - 131,\ t^3 - t^2 - 134t - 521$
13×37	$t^3 - t^2 - 160t + 467,\ t^3 - t^2 - 160t - 625$
13×43	$t^3 - t^2 - 186t + 961,\ t^3 - t^2 - 186t + 415$

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N	1	4	7	13	16	19	25	28	31	37	43	49	52	61	64	67	73	76	79	91	97
d_N	1	1	2	2	1	2	1	2	2	2	2	3	2	2	1	2	2	2	2	4	2

FIGURE 1. Coefficients of $\zeta_{\mathbb{Q}(\sqrt{-3})}(s) = \sum_N d_N N^{-s}$ for $N \equiv 1 \pmod 3$ up to 97.

Corollary 1. *If $t^3 - t^2 + at + b \in F_K$ then $a \leq 0$ and D_K divides $1 - 3a$. Fix $a \in \mathbb{Z}_{\leq 0}$. The number of polynomials of the form $t^3 - t^2 + at + b \in F_K$ for any $b \in \mathbb{Z}$ is equal to the number of integral ideals in $\mathbb{Q}(\sqrt{-3})$ with norm $N = (1-3a)D_K^{-1}$. Explicitly,*

$$\#\{\, f = t^3 - t^2 + at + b \ :\ b \in \mathbb{Z},\ K_f \cong K \,\} = \sigma_0\left(P_1\left(\frac{1-3a}{D_K}\right)\right).$$

where $\sigma_0(P)$ is the number of divisors of P and $P_1(N)$ is the largest divisor of N only divisible by primes $\equiv 1 \pmod 3$.

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2. Proof of the theorem

We use a refinement of the method from [1]. Let $\mathcal{G} = R_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-3})}\mathbb{G}_m$. Let $f \in L^1(\mathcal{G}(\mathbb{A}))$. The general Poisson summation formula — following from the classical proof for $\mathbb{Z} \subset \mathbb{R}$ — says that if $\widehat{f}\,|_{\mathcal{G}(\mathbb{Q})^\perp} \in L^1(\mathcal{G}(\mathbb{Q})^\perp)$ then

$$\int_{\mathcal{G}(\mathbb{Q})} f(xy)\, dx = \int_{\mathcal{G}(\mathbb{Q})^\perp} \widehat{f}(\chi)\chi(y)\, d\chi$$

for almost every $y \in \mathcal{G}(\mathbb{A})$ and suitably normalized Haar measure $d\chi$ on $\mathcal{G}(\mathbb{Q})^\perp$ [2, Theorem 4.4.2, p. 105]. In [1] this formula was applied with $y = 1$ and a quotient group T of $\mathcal{G}$ in place of $\mathcal{G}$ to reexpress the height zeta function of a certain compactification of T.

Let x be any trace-one normal element of the fixed tamely ramified cubic abelian field K. Here we will apply the (twisted) Poisson formula to the function

$$f(v) = H(v(K, x), -s, D_0)$$

where $H(-, s, D_0) = H(-, s)1_{D_0}(-)$ is the toric height function supported on D_0-integral points defined in [1, §3]. The only dependence on K in the Poisson formula will be in our choice of the "twisting parameter" $y \in \mathcal{G}(\mathbb{A})$. This parameter will be obtained as an approximation to the rational point $(K, x) \in T(\mathbb{Q})$.

Lemma 1. *Assume $K/\mathbb{Q}$ is tamely ramified. Then there is an element $y = y_K \in \mathcal{G}(\mathbb{A})$ and an element k of the maximal compact subgroup of $T(\mathbb{A})$ such that $(K, x) = yk$. In particular, $H((K, x), s) = H(\pi(y), s)$ where $\pi \colon \mathcal{G} \to T$ denotes the natural quotient morphism.*

Proof. Let v be a finite place of $\mathbb{Q}$. A classical theorem of Noether says that $O_v/\mathbb{Z}_v$ has a normal integral basis if and only if $K_v/\mathbb{Q}_v$ is tamely ramified. Noether's theorem implies that

$$(K_v/\mathbb{Q}_v, x) = y_v k_v$$

for some element k_v of the maximal compact subgroup of $T(\mathbb{Q}_v)$. Since $K/\mathbb{Q}$ is split at infinity, the real point $(K_\mathbb{R}/\mathbb{R}, x)$ is equal to $\pi(y_\infty)$ for some $y_\infty \in \mathcal{G}(\mathbb{R})$. Now take $y = (y_v)_v$ and $k = (k_v)$. ☐

Now we prove the main theorem. By the lemma, there is an element $y_K \in \mathcal{G}(\mathbb{A})$ such that $H(v(K, x), -s, D_0) = H(v\pi(y_K), s, D_0)$ for any $v \in \mathcal{G}(\mathbb{Q})$. The twisted Poisson formula for $f(v) = H(v(K, x), -s, D_0)$ with $y = y_K$ implies that

$$\sum_{v \in \mathcal{G}(\mathbb{Q})} H(v(K, x), -s, D_0) = \int_{\mathcal{G}(\mathbb{Q})^\perp} \widehat{f}(\chi)\chi(y_K)\, d\chi. \tag{1}$$

Let Z_K denote the left-hand side of the equality in the main theorem. There are two rational points on T corresponding to a given polynomial with Galois group $\mathbb{Z}/3\mathbb{Z}$, namely (K, x) and (K', x) where K' is the twist of the $\mathbb{Z}/3\mathbb{Z}$-algebra K by the outer automorphism of $\mathbb{Z}/3\mathbb{Z}$ [1, Example 1], so the left-hand side of (1) is equal to $2Z_K$.

To evaluate the right-hand side of (1) we make use of the calculation of $\widehat{f}(\chi)$ from [1, Prop. 4]. In addition to the new factor $\chi(y_K)$ in the integrand, there are a few differences since we are using $\mathcal{G}$ rather than T to compute the Fourier transform. At each place, we must compute the Fourier transform with respect to the split locus — the image of $\mathcal{G}(\mathbb{Q}_v) \to T(\mathbb{Q}_v)$ — rather than $T(\mathbb{Q}_v)$. At finite places $v \neq 3$ this results in the same expression for $\widehat{H_v}$ except we instead sum over the simpler sublattice (writing $E = \mathbb{Q}(\sqrt{-3})$)

$$\mathbb{Z}\langle v_1, v_2\rangle = X_*(\mathcal{G}_E)^{\Gamma(w/v)} \subset X_*(T_E)^{\Gamma(w/v)} = \mathbb{Z}\langle v_1, \omega\rangle$$

where w is any place of E over v, $\omega = \frac{2}{3}v_1 + \frac{1}{3}v_2$ and v_1, v_2 are the cocharacters of $\mathcal{G}_E$ corresponding to the two nontrivial representations of $\mathbb{Z}/3\mathbb{Z}$, and $\Gamma(w/v)$ is the decomposition group of the Galois group of $E/\mathbb{Q}$ at w. The Fourier transform at $v = \infty$ is unchanged since $X_*(\mathcal{G}_E)_{\mathbb{R}}^{\Gamma(w/\infty)} = X_*(T_E)_{\mathbb{R}}^{\Gamma(w/\infty)}$. At $v = 3$ the local height is compactly supported, and the computation in [1] shows that it does not contribute to the right-hand side of (1). This results in the following integral representation for Z_K:

$$\int_{\mathcal{G}(\mathbb{Q})^\perp} \widehat{f}(\chi)\chi(y_K)\, d\chi = \left(\frac{-1}{2\pi i}\right)\frac{s}{\pi i}\sum_{\eta} \eta^{-s}\int_{\mathbb{R}} \frac{\chi_t(\eta)^{-1}\chi_t(y_K)\, dt}{(t + \frac{s}{\pi i})(t - \frac{s}{2\pi i})}$$

where now we only sum over *split* η corresponding to the image of $\mathcal{G}(\mathbb{A}^f) \to T(\mathbb{A}^f)$; explicitly, $\eta = (m_w)_{v\neq 3,\infty}$ where

$$m_w \in \begin{cases} \mathbb{Z}_{\geq 0}\langle v_1, v_2\rangle & \text{if } w \text{ split,} \\ \mathbb{Z}_{\leq 0}\langle v_0\rangle & \text{otherwise.}\end{cases}$$

The integral can be evaluated by Cauchy's residue formula, and summing over η obtains

$$Z_K = \chi_{\frac{s}{\pi i}}(y_K)^{-1}\left(\prod_{q\equiv 2 \pmod 3}\sum_{c_q=0}^{\infty} q^{-2c_q s}\right)\left(\prod_{p\equiv 1 \pmod 3}\sum_{a_p, b_p=0}^{\infty} p^{-(a_p+b_p)s}\right).$$

This evaluates to $\chi_{\frac{s}{\pi i}}(y_K)^{-1}(1 - 3^{-s})\zeta_{\mathbb{Q}(\sqrt{-3})}$.

In particular we see that $\chi_{\frac{s}{\pi i}}(y_K)^{-1} = \lambda_K^{-s}$ where $\lambda_K = \min\{H(f)^2 : f \in F_K\}$. It remains to be seen that $\lambda_K = \sqrt{D_K}$; this follows from an elementary computation as follows. Let $\|\cdot\|$ denote the canonical norm on Minkowski space $K_{\mathbb{R}}$. Let $\|\cdot\|_1$ denote the quotient norm on $K_{\mathbb{R}}/\mathbb{R}1$ given by $\|v + \mathbb{R}1\|_1 \coloneqq \min\{\|v - r1\| : r \in \mathbb{R}\}$. If $v = (x, y, z) \in K_{\mathbb{R}}$, then calculus shows that $\|v + \mathbb{R}1\|_1^2 = \frac{1}{3}(3p_2 - 1) = \frac{2}{3}(1 - 3e_2)$, so the toric height and the quotient norm are proportional by $\sqrt{\frac{2}{3}}$. The group $\mathbb{Z}/3\mathbb{Z}$ acts unitarily on $K_{\mathbb{R}}/\mathbb{R}1$ for $\|\cdot\|_1$, so the lattice $O_K/\mathbb{Z}1$ is the standard hexagonal lattice which has covolume $\frac{\sqrt{3}}{2}\ell^2$ where ℓ is the minimal length of a nonzero lattice point. Thus $\ell = \sqrt{\frac{2}{3}}\sqrt{\lambda_K}$. On the other hand, the covolume of $O_K/\mathbb{Z}1$ is $\sqrt{3}^{-1}\cdot \mathrm{covol}(O_K \subset K_{\mathbb{R}}) = \sqrt{3}^{-1}\sqrt{D_K}$, and equating both formulas for the covolume of $O_K/\mathbb{Z}1$ shows that $\sqrt{D_K} = \lambda_K$.

References

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